Coupled Approaches for Fluid Dynamic Problems Using the PDE Framework Peano

2nd European Seminar on Coupled Problems

Philipp Neumann

30.06.2010

Contents

Motivation

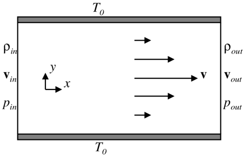
Why shall I use/ couple different fluid solvers?

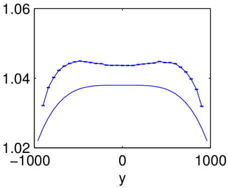
- Different features for different scenarios
- Boundary treatment [3]
- Covering different physical effects [4],[2]

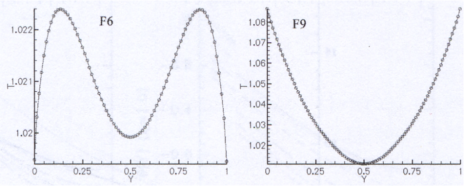
Motivation

Why shall I use/ couple different fluid solvers?

- Different features for different scenarios
- Boundary treatment [3]
- Covering different physical effects [4],[2]







The Navier-Stokes solver (NS)

- Macroscopic description of fluid dynamics:

$$\begin{aligned} \nabla \cdot u &= 0 \\ \partial_t u + u \cdot \nabla u &= -\nabla p + \frac{1}{Re} \Delta u \end{aligned}$$

- Discretisation in FE-space:

$$A\dot{u}_h + \underbrace{C(u_h)\,u_h + Du_h}_{=:F} + M^T p_h = 0 \quad (1)$$

$$Mu_h = 0 \quad (2)$$

- Solve discrete problem: Let u_h^n, p_h^n be given at timestep t^n.
  - Compute $F(u_h^n)$
  - Determine p_h^{n+1} from **Pressure Poisson equation**:

$$MA^{-1}M^T p_h^{n+1} = -MA^{-1}F$$

  - Compute u_h^{n+1} from u_h^n, p_h^{n+1} using Eq.(1)

The Lattice Boltzmann Method (LB) (1)

- Mesoscopic approach to CFD
- Discrete approximation of the Boltzmann equation
- Define $f_i(x, t) :=$ probability that fluid molecules in small surrounding of x at time t move along lattice velocity c_i ($i = 1, ..., Q$)
- Lattice velocities c_i: chosen such that molecules can either
  - rest in current cell or
  - move to a neighbouring cell
- Model of diffusive and convective transport by **Collide** and **Streaming step**

The Lattice Boltzmann Method (2)

- Collide step:
  - Compute density $\rho(x,t)$ and velocity $u(x,t)$:

$$
\begin{aligned}
\rho &:= \sum_{i=1}^{Q} f_i(x,t) \\
\rho u &:= \sum_{i=1}^{Q} f_i(x,t) c_i
\end{aligned}
$$

The Lattice Boltzmann Method (2)

- Collide step:
  - Compute density $\rho(x,t)$ and velocity $u(x,t)$:

$$\begin{aligned}\rho &:= \sum_{i=1}^{Q} f_i(x,t)\\ \rho u &:= \sum_{i=1}^{Q} f_i(x,t) c_i\end{aligned}$$

  - Compute local equilibrium distribution $f^{eq}(\rho, u)$:

$$f_i^{eq}(\rho,u) := a_i \rho \left(1 + \frac{c_i u}{c_s^2} + \frac{(c_i u)^2}{2c_s^4} - \frac{u^2}{2c_s^2}\right)$$

The Lattice Boltzmann Method (2)

- Collide step:
  - Compute density $\rho(x,t)$ and velocity $u(x,t)$:

$$
\begin{aligned}
\rho &:= \sum_{i=1}^{Q} f_i(x,t) \\
\rho u &:= \sum_{i=1}^{Q} f_i(x,t) c_i
\end{aligned}
$$

  - Compute local equilibrium distribution $f^{eq}(\rho, u)$:

$$
f_i^{eq}(\rho, u) := a_i \rho \left(1 + \frac{c_i u}{c_s^2} + \frac{(c_i u)^2}{2c_s^4} - \frac{u^2}{2c_s^2} \right)
$$

  - Relax system towards local equilibrium:

$$
f_i^*(x,t) := f_i^{eq} + \Delta_i(f^{neq}), \qquad f_i^{neq} := f_i - f_i^{eq}
$$

The Lattice Boltzmann Method (2)

- Collide step:
  - Compute density $\rho(x,t)$ and velocity $u(x,t)$:

$$\rho := \sum_{i=1}^{Q} f_i(x,t)$$

$$\rho u := \sum_{i=1}^{Q} f_i(x,t) c_i$$

  - Compute local equilibrium distribution $f^{eq}(\rho,u)$:

$$f_i^{eq}(\rho,u) := a_i \rho \left(1 + \frac{c_i u}{c_s^2} + \frac{(c_i u)^2}{2c_s^4} - \frac{u^2}{2c_s^2}\right)$$

  - Relax system towards local equilibrium:

$$f_i^*(x,t) := f_i^{eq} + \Delta_i(f^{neq}), \qquad f_i^{neq} := f_i - f_i^{eq}$$

  - BGK approximation [1] for $\Delta_i(f^{neq})$:
    $\Delta_i(f^{neq}) = \left(1 - \frac{1}{\tau}\right)\left(f_i - f_i^{eq}\right)$

The Lattice Boltzmann Method (3)

- Streaming step:

$$f_i(x + c_i dt, t + dt) := f_i^*(x, t)$$

The Lattice Boltzmann Method (3)

- Streaming step:

$$f_i(x + c_i dt, t + dt) := f_i^*(x, t)$$

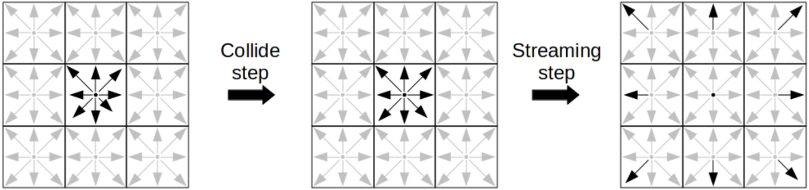


Fig.1: LB-algorithm: Pre-Collision state (left),
Post-Collision state (center), Post-Streaming state(right)

LB-NS-Coupling: Tasks

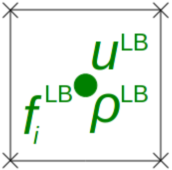
- LB $\rightarrow$ NS: Reconstruct velocity u^{NS} from LB-values

LB-NS-Coupling: Tasks

- LB $\rightarrow$ NS: Reconstruct velocity u^{NS} from LB-values
- NS $\rightarrow$ LB: Reconstruct f_i from p^{NS}, u^{NS}, $i = 1, ..., Q$

LB-NS-Coupling: Tasks

- LB $\rightarrow$ NS: Reconstruct velocity u^{NS} from LB-values
- NS $\rightarrow$ LB: Reconstruct f_i from p^{NS}, u^{NS}, $i = 1, ..., Q$
- Storage scheme:



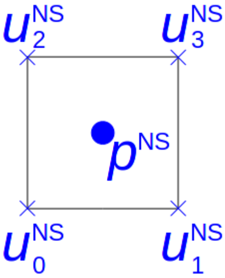


Fig.2: Storage scheme for LB (left) and NS (right)

LB-NS-Coupling: LB $\rightarrow$ NS

- *d*-linear interpolation of u^{NS} on vertices

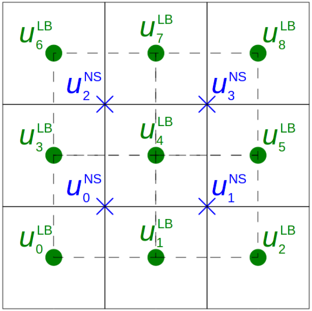


Fig.3: Interpolation from LB to NS velocities

LB-NS-Coupling: LB $\rightarrow$ NS

- *d*-linear interpolation of u^{NS} on vertices

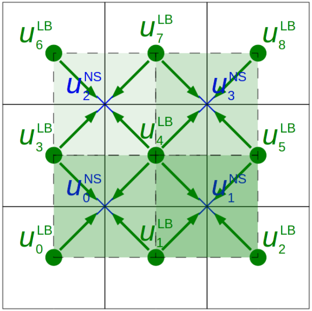


Fig.3: Interpolation from LB to NS velocities

LB-NS-Coupling: NS $\rightarrow$ LB (1)

- Problem: Reconstruct Q degrees of freedom (f_i) from $D+1$ (p^{NS}, u^{NS}), $Q > D+1$

LB-NS-Coupling: NS $\rightarrow$ LB (1)

- Problem: Reconstruct Q degrees of freedom (f_i) from $D+1$ (p^{NS}, u^{NS}), $Q > D+1$

- Approach: $f_i = f_i^{eq} + f_i^{neq}$,

$$f_i^{eq} = f_i^{eq}(\rho^{LB}, u^{LB})$$

LB-NS-Coupling: NS $\rightarrow$ LB (1)

- Problem: Reconstruct Q degrees of freedom (f_i) from $D+1$ (p^{NS}, u^{NS}), $Q > D+1$
- Approach: $f_i = f_i^{eq} + f_i^{neq}$,

$$f_i^{eq} = f_i^{eq}(\rho^{LB}, u^{LB})$$

- Therefore:
  - d-linear interpolation of u^{LB} in cell center
  - direct injection of p^{NS} onto $\rho^{LB} = \frac{1}{c_s^2} p^{LB}$

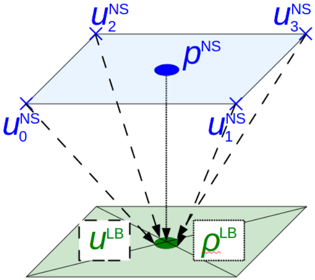


Fig.4: Interpolation from NS to LB variables

LB-NS-Coupling: NS $\longrightarrow$ LB (2)

- Open: How to reconstruct f_i^{neq}?

- Goals:

LB-NS-Coupling: NS $\longrightarrow$ LB (2)

- Open: How to reconstruct f_i^{neq}?

- Goals:

Conservation of mass	$\longrightarrow$	$\sum_i f_i^{neq}$	$= 0$

LB-NS-Coupling: NS $\longrightarrow$ LB (2)

- Open: How to reconstruct f_i^{neq}?

- Goals:

Conservation of mass	$\longrightarrow$	$\sum_i f_i^{neq}$	$= 0$
Conservation of momentum	$\longrightarrow$	$\sum_i f_i^{neq} c_{i_\alpha}$	$= 0,\ \alpha \in \{1, ..., d\}$

LB-NS-Coupling: NS $\rightarrow$ LB (2)

- Open: How to reconstruct f_i^{neq}?

- Goals:

Conservation of mass	$\rightarrow$	$\sum_i f_i^{neq}$	$= 0$
Conservation of momentum	$\rightarrow$	$\sum_i f_i^{neq} c_{i_\alpha}$	$= 0$, $\alpha \in \{1, ..., d\}$
Fit viscous stresses in collision	$\rightarrow$	$-\frac{1}{2} \sum_i \left(2 \cdot f_i^{neq} + \Delta_i(f^{neq})\right) c_{i_\alpha} c_{i_\beta}$	$= \nu \left(\frac{\partial u_\beta^{NS}}{\partial x_\alpha} + \frac{\partial u_\alpha^{NS}}{\partial x_\beta} \right)$, $\alpha, \beta \in \{1, ..., d\}$

LB-NS-Coupling: NS $\rightarrow$ LB (2)

- Open: How to reconstruct f_i^{neq}?

- Goals:

Conservation of mass	$\rightarrow$	$\sum_i f_i^{neq}$	$= 0$
Conservation of momentum	$\rightarrow$	$\sum_i f_i^{neq} c_{i_\alpha}$	$= 0,$ $\alpha \in \{1, ..., d\}$
Fit viscous stresses in collision	$\rightarrow$	$-\frac{1}{2} \sum_i \left(2 \cdot f_i^{neq} + \Delta_i(f^{neq})\right) c_{i_\alpha} c_{i_\beta}$	$= \nu \left(\frac{\partial u_\beta^{NS}}{\partial x_\alpha} + \frac{\partial u_\alpha^{NS}}{\partial x_\beta} \right),$ $\alpha, \beta \in \{1, ..., d\}$

Keep f_i^{neq} as small as possible

LB-NS-Coupling: NS $\longrightarrow$ LB (3)

- Minimise $g(f^{neq}) \in \mathcal{P}_2$ such that
  - mass and momentum are conserved
  - the viscous stresses fit
- Choices for $g : \mathbb{R}^Q \longrightarrow \mathbb{R}$:

$$g(f^{neq}) := \sum_i f_i^{neq^2} \quad \longrightarrow \quad \text{Squared } L_2\text{-norm}$$

$$g(f^{neq}) := \sum_i \left(\frac{f_i^{neq}}{f_i^{eq}} \right)^2 \quad \longrightarrow \quad \text{Squared Knudsen-norm}$$

$$g(f^{neq}) := \sum_i \left(\frac{f_i^{neq}}{a_i} \right)^2 \quad \longrightarrow \quad \text{Approx. squared Knudsen-norm}$$

$$g(f^{neq}) := \left(\sum_i \log \left(\frac{e_{f_i^{eq}}}{a_i \rho} \right) \cdot f_i^{neq} \right)^2 \quad \longrightarrow \quad \text{Squared linearised entropy deviation}$$

LB-NS-Coupling: Grid topology

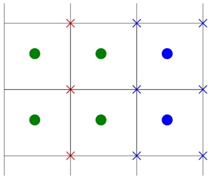


Fig.5: Grid at NS-boundary

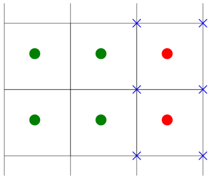


Fig.6: Grid at LB-boundary

LB-NS-Coupling: Grid topology

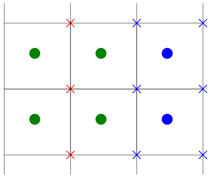


Fig.5: Grid at NS-boundary

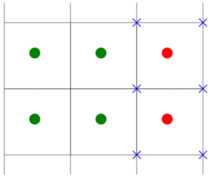


Fig.6: Grid at LB-boundary

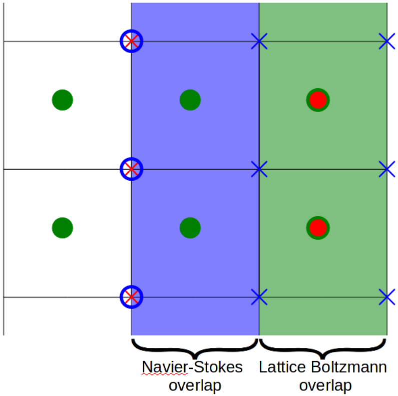


Fig.7: Overlap-Grid at LB-NS-boundary

Results: Channel flow (1)

- 2D channel flow
- Grid: 40x40 cells, LB region at walls (10 cells in y-direction), NS in the bulk and at in-/ outlet boundaries
- Parameters: $Re = 1$, $\tau = 0.56$, $dx = 1.0$, $dt = 0.01$
- Polynomial for NS-to-LB coupling: Squared Knudsen-norm

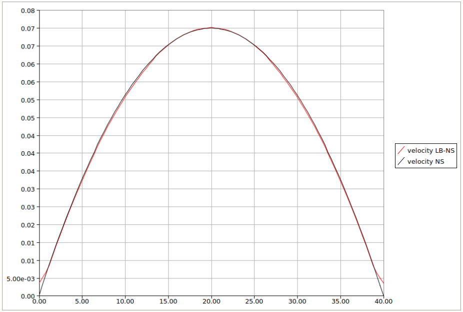


Fig.8: Velocity profile NS vs. LB-NS

Results: Channel flow (2)

- 2D channel flow
- Grid: 40x40 cells, LB region at walls (10 cells in y-direction), NS in the bulk and at in-/ outlet boundaries
- Parameters: $Re = 1$, $\tau = 0.56$, $dx = 1.0$, $dt = 0.01$
- Polynomial for NS-to-LB coupling: Squared Knudsen-norm

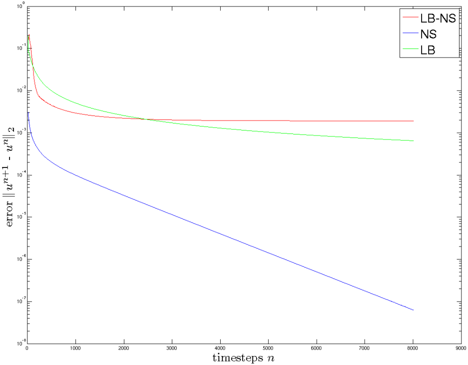


Fig.9: Convergence NS vs. LB vs. LB-NS

Results: Channel flow (3)

- 2D channel flow
- Grid: 20x20 cells, LB region in bulk (8x8 cells in domain center)
- Parameters: $Re = 1$, $\tau = 0.51$, $dx = 1.0$, $dt = 0.01$
- Polynomial for NS-to-LB coupling: Squared Knudsen-norm

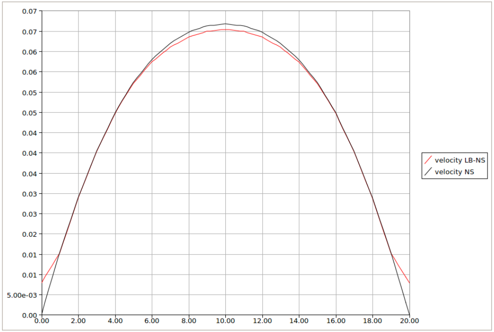


Fig.10: Velocity profile NS vs. LB-NS

Results: Channel flow (3)

- 2D channel flow
- Grid: 20x20 cells, LB region in bulk (8x8 cells in domain center)
- Parameters: $Re = 1$, $\tau = 0.51$, $dx = 1.0$, $dt = 0.01$
- Polynomial for NS-to-LB coupling: Squared Knudsen-norm

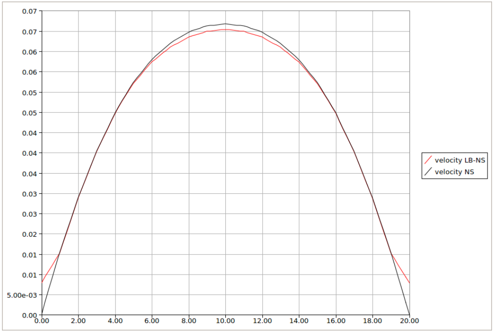


Fig.10: Velocity profile NS vs. LB-NS

$\rightarrow$ Negative distribution functions for **pure** LB simulation!

Conclusion & Outlook

Conclusion:

- New coupling strategy between mesoscopic LB formulation and macroscopic NS equations
- Convergence rate
- Stability enhancement

Conclusion & Outlook

Conclusion:

- New coupling strategy between mesoscopic LB formulation and macroscopic NS equations
- Convergence rate
- Stability enhancement

Outlook:

- 3D-scenarios
- Adaptive simulations ($\rightarrow$ Peano Framework)
- Coupled multiscale scenarios with different behaviours on different scales

References

P. L. Bhatnagar, E. P. Gross, and M. Krook.
A Model for Collision Processes in Gases. I. Small Amplitude Processes in Charged and Neutral One-Component Systems.
Phys. Rev., 94(3):511–525, May 1954.

George Karniadakis, Ali Beskok, and Narayan Aluru.
Microflows and Nanoflows.
2005.

J. Latt, B. Chopard, and P. Albuquerque.
Spatial Coupling of a Lattice Boltzmann Fluid Model with a Finite Difference Navier-Stokes Solver.
2005.
http://www.citebase.org/abstract?id=oai:arXiv.org:physics/0511243.

Y. Zheng, A.L. Garcia, and B.J. Alder.
Comparison of Kinetic Theory and Hydrodynamics for Poiseuille Flow.
Proceedings of 23rd International Symposium on Rarefied Gas Dynamics, 2003.